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Some Basic Theorems On Vector Space

Deepak

Previous Student (Open Researcher), M.Phil Mathematics

Maharshi Dayanand University, Rohtak, Haryana, India

Email ID: deepak20dalal19@gmail.com

Abstract

A non-empty set V *along with two binary operations '*$+$*' (Internal binary operation) & '*$.$*' (External binary operation) is denoted by* $V(F)$ *where* $(V,+,.)$ *is the vector space and* $(F,+,.)$ *is the field which is known as scalars field, is called a vector space or linear vector space over the given field* F *if it satisfy the axioms given below:*

i. *The algebraic structure* $(V,+)$ *is an abeliangroup.*
ii. *For any scalar* $s \in F$ *and vector* $x \in V$ *we have* $s.x \in V$.
iii. $\forall s, s_1, s_2 \in V$ *and* $\forall x, y \in V$, *then*
   (a) $(s_1 + s_2).x = s_1.x + s_2.x$
   (b) $s.(x + y) = s.x + s.y$
   (c) $s_1(s_2x) = (s_1s_2).$
   (d) $u.y = y$; u *be the unity of the field F.*

By a space in this theory we mean a vector space or the linear space. Here we take the standard examples of vector spaces as below Let us consider the standard fields of complex numbers, real numbers and rational numbers denoted by $\mathbb{C}, \mathbb{R}$ *and* $\mathbb{Q}$ *respectively. Then* $\mathbb{C}(\mathbb{C})$, $\mathbb{C}(\mathbb{R})$, $\mathbb{C}(\mathbb{Q})$, $\mathbb{R}(\mathbb{R})$, $\mathbb{R}(\mathbb{Q})$ & $Q(Q)$ *are the vector spaces.*

Keywords: Vector Space.

Notation:

1-Space ($\Re$) = $\{x \mid x$ is a real number $\}$;

2-Space ($\Re^2$) = $\{(x, y) \mid x, y$ are real numbers$\}$;

3-Space ($\Re^3$) = $\{(x, y, z) \mid x, y, z$ are real numbers$\}$;

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4-Space ($\Re^4$) $=\{(x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4$ are real numbers$\}$;

…

***n*-Space ($\Re^n$)** $=\{(x_1, x_2, ..., x_n) \mid x_1, x_2, ..., x_n$ are real numbers$\}$

The elements of $\Re^n$ are called **points** or **vectors**. They are usually denoted by boldface letters as

$$\mathbf{x} = (x_1, x_2, ..., x_n) = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}_{nx1} \leftarrow n-\textbf{tuple} \text{ of real numbers}$$

The *i*th entry of the vector $\mathbf{x} = (x_1, x_2, ...x_i, ..., x_n)$ is called its ***i*th coordinate** or its ***i*th component**.

The **zero vector** in $\Re^n$ is

$$\mathbf{0} = (0, 0, \ldots, 0).$$

If $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ are vectors in $\Re^n$**,** then their **sum** is defined as the vector

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n).$$

If *c* is a **scalar** (a real number), then the **scalar multiple** of the vector **x** by the scalar *c*, denoted by *c***x,** is the vector

$$c\mathbf{x} = (cx_1, cx_2, ..., cx_n)$$

Note:

$$(-1)\mathbf{x} = -\mathbf{x} = (-x_1, -x_2, ..., -x_n)$$

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Vector Space: Let V be a set vectors in which the operations of sum of vectors and of scalar multiplication are defined (that is, given vectors **x** and **y** in V and a scalar c, the vectors **x** + **y** and c**x** are also in V - in this case V is said to be **closed** under vector addition and multiplication by scalars). Then with these operations V is called a **vector space** provided that - given any vectors **x, y,** and **z** in V and any scalars a and b - the following properties are true:

a. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ **(commutativity)**
b. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ **(associativity)**
c. $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$ **(zero element)**
d. $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$ **(additive inverse)**
e. $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$ **(distributivity)**
f. $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{y}$
g. $a(b\mathbf{x}) = (ab)\mathbf{x}$
h. $(1)\mathbf{x} = \mathbf{x}$

Theorem: The n-space $\Re^n$ is a **vector space.**

Let W be a nonempty subset of the vector space V. If W is a vector space with the operations of addition and scalar multiplication as defined in V, then W is a **subspace** of V.

Examples:

1. $W = \{\mathbf{0}\}$ is a subspace of $\Re^n$ (called the **zero subspace).**

2. $W = \Re^n$ is a subspace of $\Re^n$ (also called the **improper subspace).**

(all other subspaces of $\Re^n$ are called **proper subspaces**)

Theorem: (Conditions for a subspace)

The nonempty subset W of the vector space V is a subspace of V if and only if it satisfies the following conditions:

a. **0** is in W;

a. If **x** and **y** are vectors in W, then **x** + **y** is also in W;
b. If **x** is in W and c is a scalar, then the vector c**x** is also in W.

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Theorem: (Solution subspaces)

If A is an m x n matrix of constants, then the solution set of the homogeneous linear system

$$A\mathbf{x} = \mathbf{0}$$

is a subspace of $\Re^n$.

Example: Find two solution vectors **u** and **v** for the following homogeneous system such that the solution space is the set of all linear combinations of the form a**u** + b**v**:

$$\begin{aligned} 2x + 4y \quad - 5z + 3w &= 0 \\ 3x + 6y \quad - 7z + 4w &= 0 \\ 5x + 10y - 11z + 6w &= 0 \end{aligned}$$

We reduce the coefficient matrix to echelon form by applying the following sequence of EROs: $-3R_1 + 2R_2,\ -5R_1 + 2R_3,\ -3R_2 + R_3$.

The echelon matrix we obtain is

$$\begin{bmatrix} 2 & 4 & -5 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence x and z are the **leading variables**, and y and w are the **free variables**. Back substitution yields the general solution

$$y = a, w = b,\ z = b, x = -2a + b$$

Thus the general solution vector of the system has the form

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2a + b \\ a \\ b \\ b \end{bmatrix} = a\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = a\mathbf{u} + b\mathbf{v}$$

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where **u** = (-2, 1, 0, 0) and **v** = (1, 0, 1, 1).

The solution space of the system is completely determined by the vectors **u** and **v** by the formula $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$.

The vector **y** is called a **linear combination** of the vectors $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}$ provided that there exists scalars $c_1, c_2, ..., c_n$ such that

$$\mathbf{y} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + ... + c_n\mathbf{x_n}$$

Let $S = \{\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}\}$ be a set of vectors in the vector space *V*. The set of all linear combinations of $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}$ is called the **span** of the set *S*, denoted by span(*S*) or span($\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}$).

Theorem: span(*S*) is a subspace of *V*.

The set $S = \{\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}\}$ of vectors in the vector space *V* is a **spanning set** for *V* provided that every vector in *V* is a linear combination of the vectors in *S*.

The set of vectors $S = \{\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}\}$ in a vector space *V* is said to be **linearly independent** provided that the equation

$$c_1\mathbf{x_1} + c_2\mathbf{x_2} + ... + c_n\mathbf{x_n} = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = ... = c_n = 0$

Example : The **standard unit vectors** in $\Re^n$, viz.,

$$e_1 = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}, \quad e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \quad \cdots \quad e_n = \begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix},$$

are linearly independent.

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Note:

Any **subset of a linearly independent** set is a **linearly independent set**.

The coefficients in a linear combination of the vectors in a linearly independent set are unique.

A set of vectors is called **linearly dependent** if it is not linearly independent.

Example : The vectors **u** = (1, -1, 0), **v** = (1, 3, -1), and **w** = (5, 3, -2) are linearly dependent since 3**u** + 2**v** - **w** = 0.

Exercise: Determine whether the following vectors in $\Re^4$ are linearly dependent or independent.

$$(1, 3, -1, 4), \; (3, 8, -5, 7), \; (2, 9, 4, 23).$$

The vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ are linearly dependent if and only if at least one of them is a linear combination of the others.

Theorem: The n vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ in $\Re^n$ are linearly independent if and only if the

n x n matrix

$$A = [\; \mathbf{x}_1 \;\; \mathbf{x}_2 \;\; \ldots \;\; \mathbf{x}_n \;]$$

with the vectors as columns has nonzero determinant.

Theorem: Consider k vectors in $\Re^n$, with $k < n$. Let

$$A = [\; \mathbf{x}_1 \;\; \mathbf{x}_2 \;\; \ldots \;\; \mathbf{x}_k \;]$$

be the n x k matrix having the k vectors as columns. Then the vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$

are linearly independent if and only if some k x k submatrix of A has nonzero determinant.

A finite set S of vectors in a vector space V is called a **basis** for V provided that

(a) The vectors in S are linearly independent;
(b) The vectors in S span V.

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Example : The set of **standard unit vectors** in $\Re^n$ **, viz.,**

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

form the **standard basis** for $\Re^n$.

Note: Any set of n linearly independent vectors in $\Re^n$ is a **basis** for $\Re^n$ **.**

Example : u = (-2, 1, 0, 0) and **v** = (1, 0, 1, 1) is a basis of the solution space of the homogeneous system given in **First Example**. The dimension of the solution space is 2.

Theorem: Let $S = \{\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}\}$ be a basis for the vector space V. Then any set of more than n vectors in V is linearly dependent**.**

Theorem: Any two bases of a vector space consist of the same number of vectors.

A vector space V is called **finite dimensional** if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the **dimension** of V and is denoted by **dim(*V*)**.

A vector space that is not finite dimensional is called **infinite-dimensional.**

Let $A = [a_{ij}]_{mxn}$ be a matrix. The row vectors of A are the m vectors in $\Re^n$ given by

$$\begin{aligned} \mathbf{r}_1 &= (a_{11}, a_{12}, \ldots, a_{1n}) \\ \mathbf{r_2} &= (a_{21}, a_{22}, \ldots, a_{2n}) \\ &\vdots \\ \mathbf{r_m} &= (a_{m1}, a_{m2}, \ldots, a_{mn}) \end{aligned}$$

The subspace of $\Re^n$ spanned by the m row vectors $\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_m}$ is called the **row space** of the matrix A and is denoted by **Row(*A*)**.

The dimension of the row space Row(A) is called the **row rank** of the matrix A.

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Theorem: The nonzero row vectors of an echelon matrix are linearly independent and therefore form a basis for its row space Row(A).

Theorem: If two matrices are equivalent, then they have the same row space.

Note: ***To find a basis for the row space of a matrix, reduce the matrix to echelon form. Then the nonzero row vectors of the echelon matrix form a basis for the row space.***

Let $A=[a_{ij}]_{mxn}$ be a matrix. The column vectors of A are the n vectors in $\Re^m$ given by

$$\mathbf{c_1}=\begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix},\quad \mathbf{c_2}=\begin{bmatrix}a_{12},\\a_{22}\\\vdots\\a_{m2}\end{bmatrix},\quad\cdots\quad \mathbf{c_n}=\begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix}$$

The subspace of $\Re^m$ spanned by the n column vectors $\mathbf{c_1},\mathbf{c_2},\ldots,\mathbf{c_n}$ is called the **column space** of the matrix A and is denoted by **Col(*A*)**.

The dimension of the row space Col(A) is called the **column rank** of the matrix A.

Note: ***To find a basis for the column space of a given matrix, reduce the matrix to echelon form. Then the column vectors of the given matrix that correspond to the pivot columns of the echelon matrix form a basis for the column space.***

Exercise: Find a basis for the row space of the following matrix and state the dimension (row rank) of the row space. Also find a basis for the column space of the given matrix and state the dimension(column rank) of the column space.

$$A=\begin{bmatrix}1 & -1 & -1\\4 & -3 & -1\\3 & -1 & 3\end{bmatrix}$$

Theorem: The row rank and the column rank of any matrix are equal.

The common value of the row rank and column rank of a matrix is called the **rank** of the matrix.

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The **solution space** of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the **null space** of A, denoted by **Null(A)**

Note:

1. If A is an m x n matrix, then Null(A) and Row(A) are subspaces of $\Re^n$, whereas Col(A) is a subspace of $\Re^m$.

2. **rank(A) + dim Null(A) = n.**

Application: Consider a homogeneous system of m linear equations in n unknowns

($m \leq n$). If the m x n coefficient matrix has rank r, (so r out of the m equations are **irredundant**), the system has $n - r$ linearly independent solutions.

Theorem: The nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the vector $\mathbf{b}$ is in the column space of A.

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